Grade 7/8 Math Circles

February 9, 2022

Ancient Mathematics - Solutions

Exercise Solutions

Exercise 1

Determine the value each Roman Numeral represents.

(a) DC

(b) CLXVII

(c) XCIV

Exercise 1 Solution

(a) $\mathrm{D} + \mathrm{C} = 500 + 100 = 600$

(b) $\mathrm{C} + \mathrm{L} + \mathrm{X} + \mathrm{V} + \mathrm{I} + \mathrm{I} = 100 + 50 + 10 + 5 + 1 + 1 = 167$

(c) $(\mathrm{C} - \mathrm{X}) + (\mathrm{V} - \mathrm{I}) = (100 - 10) + (5 - 1) = 90 + 4 = 94$

Exercise 2

Express each using Roman Numerals.

(a) 75

(b) 2022

(c) Your age

Exercise 2 Solution

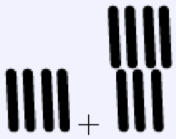
(a) LXXV

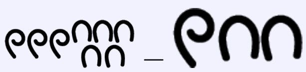
(b) MMXXII

(c) Answers may vary

Exercise 3

Evaluate each expression. Give your final answer as hieroglyphs.

(a) 

(b)  (that's a minus)

Exercise 3 Solution

(a) 

(b) 

Exercise 4

Evaluate the following using the Egyptian Multiplication technique.

(a) 21×18

(b) 

Exercise 4 Solution

(a) 21×18

21	18
1	18
~~2~~	~~36~~
4	72
~~8~~	~~144~~
16	288

$21 \times 18 = 18 + 72 + 288 = 378$

(b) 127×13

33	24
1	24
~~2~~	~~48~~
~~4~~	~~96~~
~~8~~	~~192~~
~~16~~	~~384~~
32	768

$33 \times 24 = 24 + 768 = 792$

Exercise 5

Evaluate $127 \div 13$ using the Egyptian Division technique.

Exercise 5 Solution

127	13
1	13
~~2~~	~~26~~
~~4~~	~~52~~
8	104

The remainder is $127 - 104 - 13 = 10$.
$127 \div 13 = 1 + 8 = 9$ with remainder 10.

Exercise 6

Suppose you must divide 5 loaves of bread among 8 workers. You must distribute pieces of the bread equally, so we are left with the expression $\frac{5}{8}$. As a decimal this is 0.625, but how much is 62.5% of a loaf of bread?
How would you divide 5 loaves of bread among 8 people equally? Keep in mind that the workers must be convinced that they are not getting less than their coworkers.

Exercise 6 Solution

There are many approaches but using unit fractions as a guide we can come up with a practical and mathematical solution.
Set aside 1 loaf of bread. With 4 loaves, it's pretty straightforward that each person can get

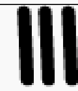
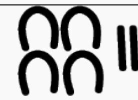
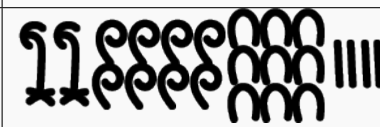
half of one loaf. So cut the four loaves into halves and distribute one piece to each worker. Going back to our remaining loaf, we can cut it into 8 equal pieces and distribute a piece to each person.
Mathematically, $\frac{5}{8} = \frac{1}{2} + \frac{1}{8}$. So to answer the question, give each worker half a loaf and $\frac{1}{8}$ of another.

Problem Set Solutions

1. Complete the chart of values. Some sections may not have an answer.

Hindu-Arabic Numerals	Roman Numerals	Egyptian Hieroglyphs
3		
		
	CCCLXXVII	
2894		
	CXLI	
		
0		

Solution:

Hindu-Arabic Numerals	Roman Numerals	Egyptian Hieroglyphs
3	III	
42	XLII	
377	CCCLXXVII	
2894	MMDCCCXCIV	
141	CXLI	
1/7	none	
0	none	none

2. What counting system do Roman Numerals follow? What about Ancient Egyptians? Are there any aspects about Egyptian Multiplication and Division that remind you of another counting system?

   For a refresher, the lesson on Counting Systems from Fall 2021 can be found here.

Solution: Both Roman Numerals and Egyptian numbers follow the **Decimal** counting system (also known as base-10). One could argue that Roman Numerals use base-5 because of the symbols for 5 (V).
In Egyptian Multiplication/Division, the doubling, or powers of 2, may remind you of forming Binary numbers.

3. Evaluate the following using the Egyptian Multiplication technique.
   (a) 17×11
   (b) 61×23
   (c) 256×7

Solution:

(a) 17×11

17	11
1	11
~~2~~	~~22~~
~~4~~	~~44~~
~~8~~	~~88~~
16	176

So $17 \times 11 = 11 + 176 = 187$.

(b) 61×23

61	23
1	23
~~2~~	~~46~~
4	92
8	184
16	368
32	736

So $61 \times 23 = 23 + 92 + 184 + 368 + 736 = 1403$.

(c) 256×7

256	7
~~1~~	~~7~~
~~2~~	~~14~~
~~4~~	~~28~~
~~8~~	~~56~~
~~16~~	~~112~~
~~32~~	~~224~~
~~64~~	~~448~~
~~128~~	~~896~~
256	1792

So $256 \times 7 = 1792$.

4. Evaluate the following using the Egyptian Division technique.
   (a) $45 \div 18$
   (b) $256 \div 7$
   (c) $1024 \div 30$

Solution:

(a) $45 \div 18$

45	18
~~1~~	~~18~~
2	36

$45 \div 18 = 2$ with remainder $45 - 36 = 9$.

(b) $256 \div 7$

256	7
~~1~~	~~7~~
~~2~~	~~14~~
4	28
~~8~~	~~56~~
~~16~~	~~112~~
32	224

$256 \div 7 = 32 + 4 = 36$ with remainder $256 - 224 - 28 = 4$.

(c) $1024 \div 30$

1024	30
~~1~~	~~30~~
2	60
~~4~~	~~120~~
~~8~~	~~240~~
~~16~~	~~480~~
32	960

$1024 \div 30 = 32 + 2 = 34$ with remainder $1024 - 960 - 60 = 4$.

5. Answer the following, using Exercise 6 of the lesson as a guide.
   (a) Describe how you could divide 13 loaves of bread among 12 people.
   (b) One of the aforementioned 12 people cannot eat bread due to allergies. How could you divide 13 loaves of bread among 11 people?

Solution:

(a) Each person gets 1 loaf at first. Cut the last loaf into 12 equal pieces and distrbute.

(b) Let Jerry be the name of the person who cannot eat bread. Divide the loaves as described in part (a). Then take Jerry's loaf and divide that into 11 equal pieces and distribute among the remaining 11. Then take his twelvth of a loaf and divide *that* into 11 equal pieces and distribute among the remaining 11.

In Ancient Egypt however, you probably still want to feed Jerry. Go to the market and trade a loaf of bread for something he can eat.

If you have a different solution, post it on Piazza!

6. (a) Can you express $\frac{1}{21}$ as a sum of two of the same unit fraction?

(b) How can we express $\frac{1}{n}$ as a sum of two unit fractions for any positive integer n?

Solution:

(a) $\frac{1}{21} = \frac{2}{42} = \frac{1}{42} + \frac{1}{42}$

(b) $\frac{1}{n} = \frac{1}{2n} + \frac{1}{2n}$

7. As mentioned, we want to find reliable ways to rewrite any fraction as sums of unit fractions. We'll begin by building our intuition of unit fractions. This question shows us that we can rewrite any unit fraction as a sum of two distinct unit fractions.

(a) Evaluate the following using a method of your choice.

i. $\frac{1}{3} + \frac{1}{6} =$

ii. $\frac{1}{4} + \frac{1}{12} =$

iii. $\frac{1}{5} + \frac{1}{20} =$

iv. $\frac{1}{6} + \frac{1}{30} =$

(b) Do you notice a pattern? Can you use this pattern to express $\frac{1}{22}$ as a sum of two unit fractions? (Hint: How do the denominators all relate to each other?)

(c) In words, describe the generalized pattern for expressing the unit fraction $\frac{1}{n}$ as a sum of two distinct unit fractions. Or, complete the formula given below to describe the pattern algebraically.

$$\frac{1}{n} = \frac{1}{(\qquad)} + \frac{1}{(\qquad)(\qquad)}$$

(d) Use the formula to express $\frac{1}{2022}$ as a sum of two unit fractions.

Solution:

(a) i. $\frac{1}{2}$ ii. $\frac{1}{3}$ iii. $\frac{1}{4}$ iv. $\frac{1}{5}$

(b) Looking at the denominators, we see that $6 \div 3 = 2, 12 \div 4 = 3, 20 \div 5 = 4$, and $30 \div 6 = 5$. Also notice that the smaller denominators have a difference of 1 (e.g. $3 - 2 = 1, 4 - 3 = 1, \ldots$). We predict

$$\frac{1}{22} = \frac{1}{23} + \frac{1}{506}$$

(c)

$$\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)}$$

(d) $\frac{1}{2022} = \frac{1}{2023} + \frac{1}{(2022)(2023)} = \frac{1}{2023} + \frac{1}{4\ 090\ 506}$

8. This question deals with unit fractions with even denominators. Using your findings from the previous question as a guide, complete the following.

   (a) Evaluate the following using a method of your choice.

   i. $\frac{1}{6} + \frac{1}{12} =$

   ii. $\frac{1}{8} + \frac{1}{24} =$

   iii. $\frac{1}{10} + \frac{1}{40} =$

   (b) Do you notice a pattern? Can you use this pattern to express $\frac{1}{12}$ as a sum of unit fractions?

   (c) Complete the formula for expressing the unit fractions $\frac{1}{n}$ as a sum of unit fractions where n is an even number. (Hint: It may be helpful to say that $n = 2m$, where m is an integer)

$$\frac{1}{n} = \frac{1}{(\qquad)} + \frac{1}{(\qquad)(\qquad)}$$

Solution:

(a) i. $\frac{1}{4}$ ii. $\frac{1}{6}$ iii. $\frac{1}{8}$

(b) Like before, look at the denominators. If we wanted to predict the sum for $\frac{1}{12}$, we can expect that one of the unit fractions in its sum has 14 as a denominator (there is a +2 pattern like the +1 pattern from Question 7). The other unit fraction can be found by multiplying 12×14 and then dividing that by 2. We get

$$\frac{1}{12} = \frac{1}{14} + \frac{1}{84}$$

(c)

$$\frac{1}{n} = \frac{1}{2m} = \frac{1}{2m+2} + \frac{1}{2m(m+1)}$$

9. Recall from the lesson that $\frac{2}{3} = \frac{1}{3} + \frac{1}{3}$ is not a valid expression of an Egyptian fraction (since the two unit fractions aren't distinct).

   (a) Express $\frac{2}{3}$ as a sum of three distinct unit fractions. (Hint: Question 7 might be helpful)

   (b) Express $\frac{2}{3}$ as a sum of two distinct unit fractions.

Solution:

(a) Answers may vary. In Question 7(a) ii., we found that $\frac{1}{3} = \frac{1}{4} + \frac{1}{12}$. If we substitute this sum in for one of the $\frac{1}{3}$ in the given expression we get

$$\frac{2}{3} = \frac{1}{3} + \frac{1}{4} + \frac{1}{12}$$

(b) One way would be

$$\frac{2}{3} = \frac{1}{2} + \frac{1}{6}$$

10. **Challenge:** Show that we can write $\frac{1}{2}$ as a sum of infinitely many unit fractions.

Solution: Many possible solutions exist, but here is one. Using the formula from Question 7, we found

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{6}$$

We can apply the formula again to $\frac{1}{3}$ to get.

$$\frac{1}{2} = \left(\frac{1}{4} + \frac{1}{12}\right) + \frac{1}{6}$$

Furthemore we can apply the formula on all of the fractions above since they are also unit fractions.

$$\frac{1}{2} = \left(\left(\frac{1}{5} + \frac{1}{20}\right) + \left(\frac{1}{13} + \frac{1}{156}\right)\right) + \left(\frac{1}{7} + \frac{1}{42}\right)$$

We could continue this process infinitely many times.